An introduction to Eulerian posets and the cd-index.

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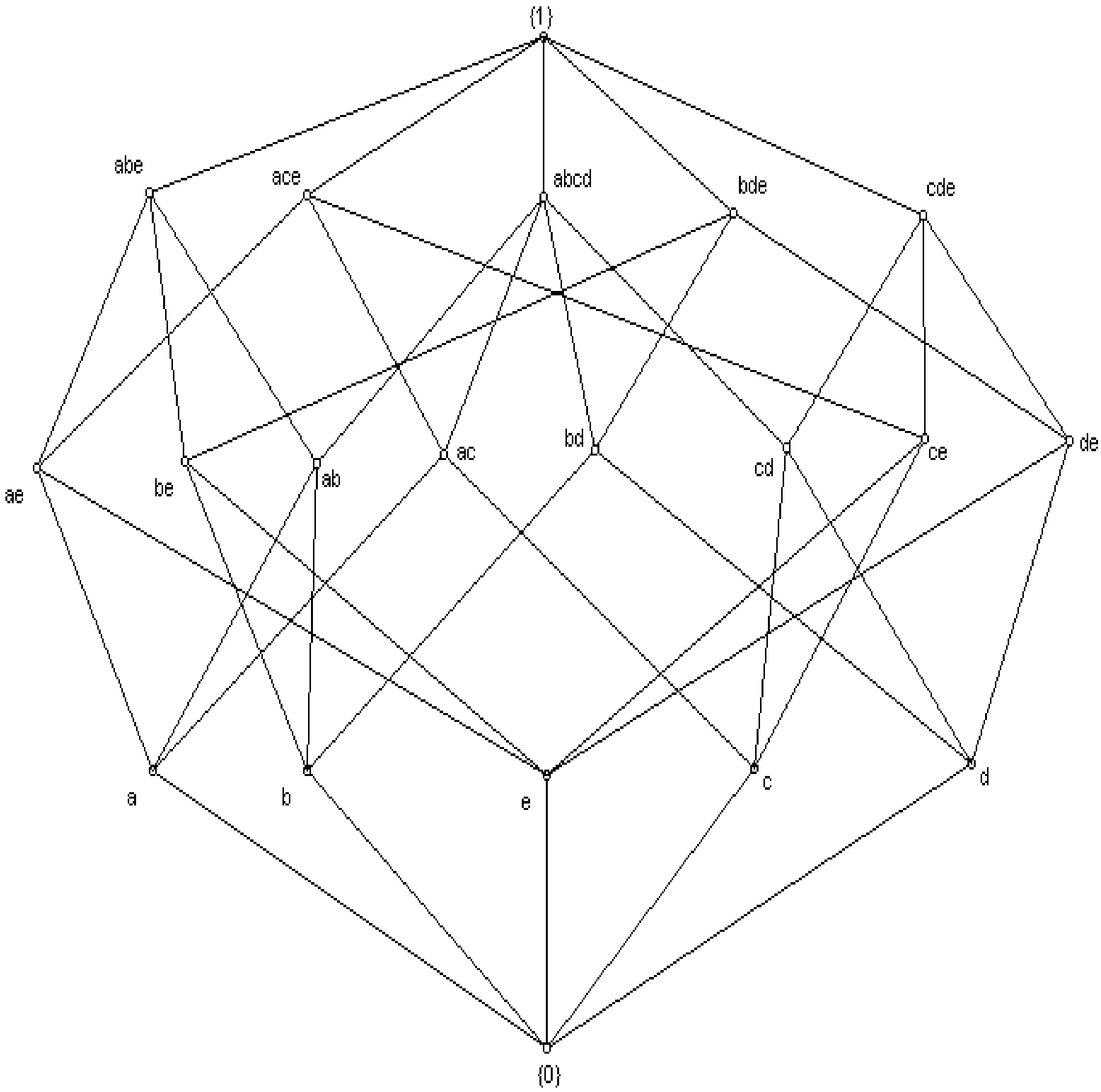
Abstract: I introduce through Eulerian posets, specifically face lattices of a convex polytope, the cd-index. Conjectured by Fine and introduced in ”A new index for polytopes” by Bayer and Klapper. The cd-index has proven to be a valuable tool in compressing the flag h-vector of a polytope from 2^n elements of information into the n^{th} Fibonacci number of elements.

Eulerian Poset

An Eulerian poset is a finite graded poset with $\hat{0}$ and $\hat{1}$ such that every interval of length at least one has the same number of elements of odd rank as of even rank.

The face lattice of a convex polytope is Eulerian.

Example of an Eulerian poset



$\mathcal{L}(\mathbf{BiPyr}(\square))$

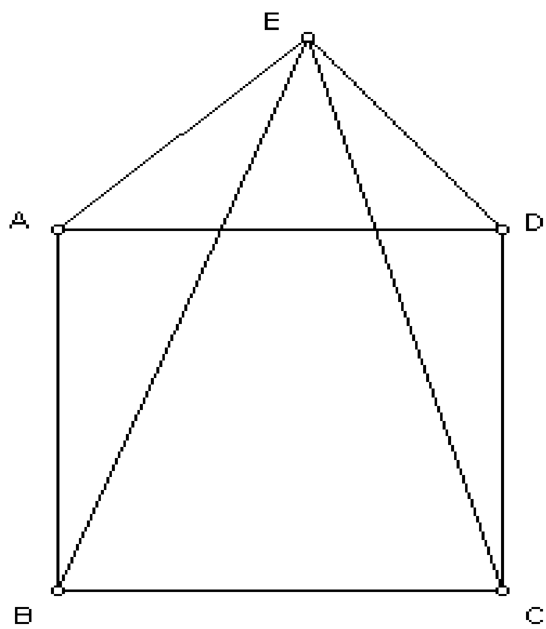
Semi-Eulerian Poset

A poset P is Semi-Eulerian if $P \setminus \{\hat{0}, \hat{1}\}$ is Eulerian.

Half-Eulerian Poset

A Half-Eulerian poset is a graded poset whose horizontal double is Eulerian. The horizontal double of a poset P is the poset obtained from P by replacing every $x \in P \setminus \{\hat{0}, \hat{1}\}$ with two elements x_1, x_2, such that $\hat{0}$ and $\hat{1}$ remain the minimum and maximum elements of the poset, and $x_i < y_j$ if and only if $x < y$ in P. (In the Hasse diagram of P, every edge is replaced by $\bowtie$).[Bayer-Hetyei, "Flag vectors of Eulerian partially ordered sets."]

Egyptian Pyramid



$$\boxed{S \subseteq \{0, 1, \ldots, d-1\}}$$

dim of faces $\in \{0, 1, 2\}$ and in general $\{0, 1, \ldots, d-1\}$.

Let $S \subseteq \{0, 1, \ldots, d-1\}$ say $S = \{i_1, i_2, \ldots, i_k\}$.

$$\mathbf{S} \subseteq \{0,1,2\}$$

S	f_S	h_S	w_S
$\emptyset$			
$\{0\}$			
$\{1\}$			
$\{2\}$			
$\{0,1\}$			
$\{0,2\}$			
$\{1,2\}$			
$\{0,1,2\}$			

flag f-vector

$f_s = \#\{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k\}$ with dim(F_i)=i.

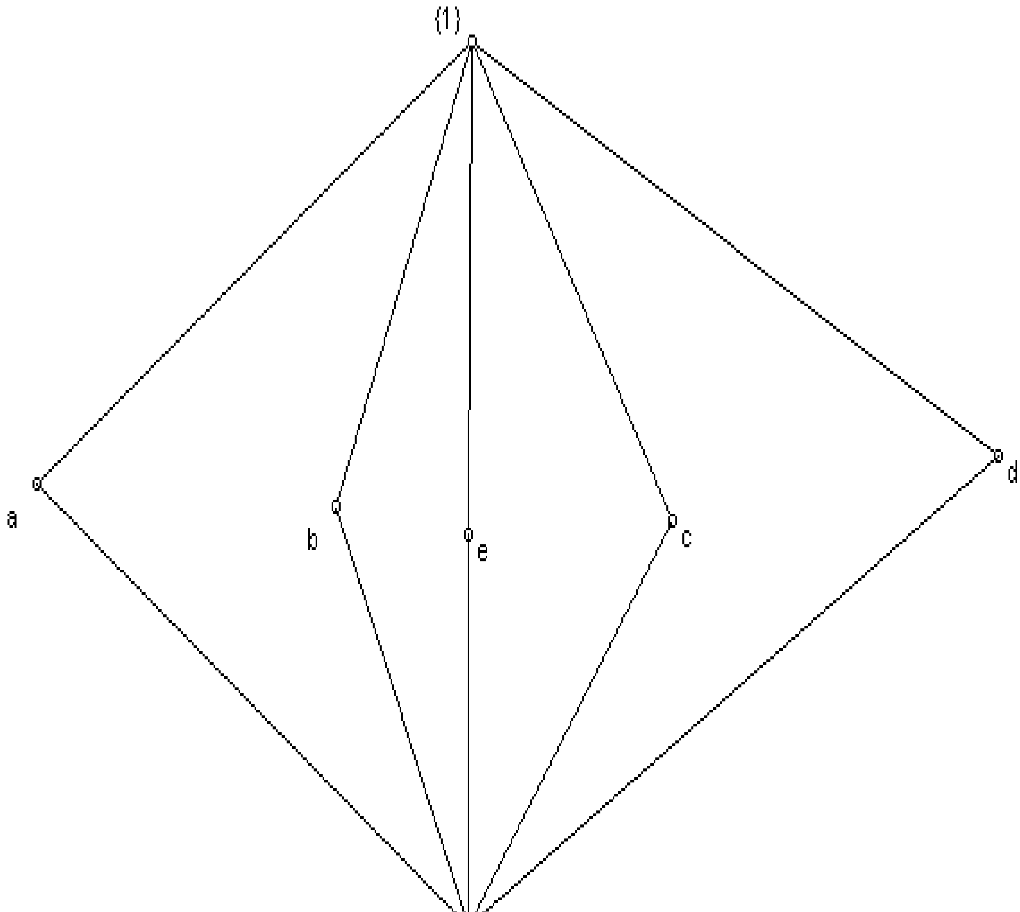
flag f-vector

S	f_S	h_S	w_S
$\emptyset$	1		
$\{0\}$	5		
$\{1\}$	8		
$\{2\}$	5		
$\{0,1\}$	16		
$\{0,2\}$	16		
$\{1,2\}$	16		
$\{0,1,2\}$	32		

flag f-vector from Eulerian Posets

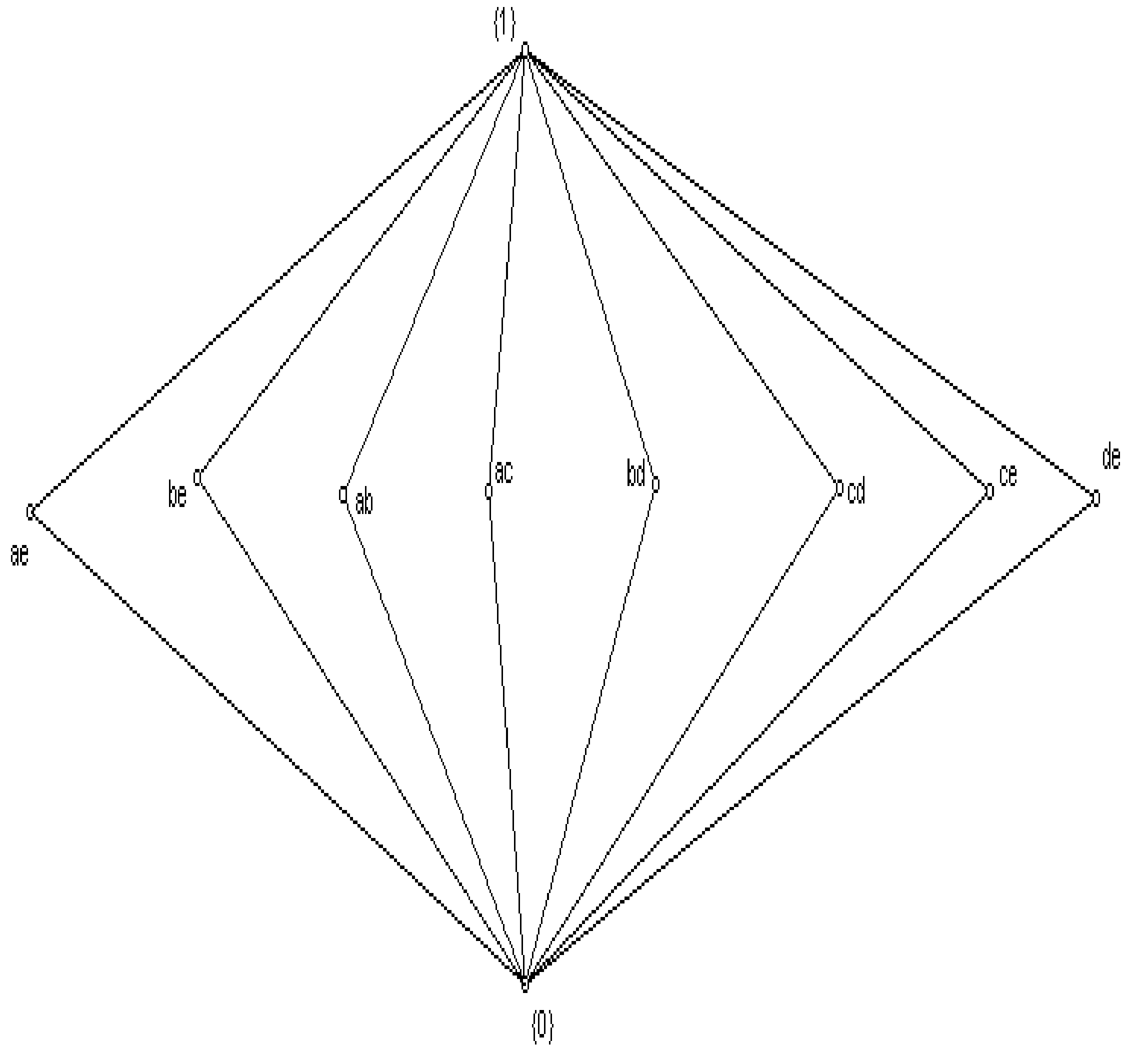
Let P be a finite graded poset of rank $n+1$ with $\hat{0}$ and $\hat{1}$. (We always assume $n \geq 0$, so $\hat{0} < \hat{1}$.) Let ρ denote the rank function of P. Thus $P = P_0 \cup P_1 \cup \cdot \cup P_{n+1}$ (disjoint union), where $x \in P_i$ if and only if $\rho(x) = i$. Every maximal chain of P has the form $\hat{0} = x_0 < x_1 < \cdot < x_{n+1} = \hat{1}$ with $\rho(x_i) = i$. Let $S \subseteq [n] = \{1, 2, \ldots, n\}$, and let P_S denote the S-rank selected subposet of P (with $\hat{0}$ and $\hat{1}$), i.e.,
$P_S = \{x \in P : \ \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}$. Denote by $\alpha(S) = \alpha_P(S)$ the number of maximal chains of P_S. $\alpha(S)$ is equivalent to the flag f-vector.

$$P_{\{1\}} = f_{\{0\}}$$



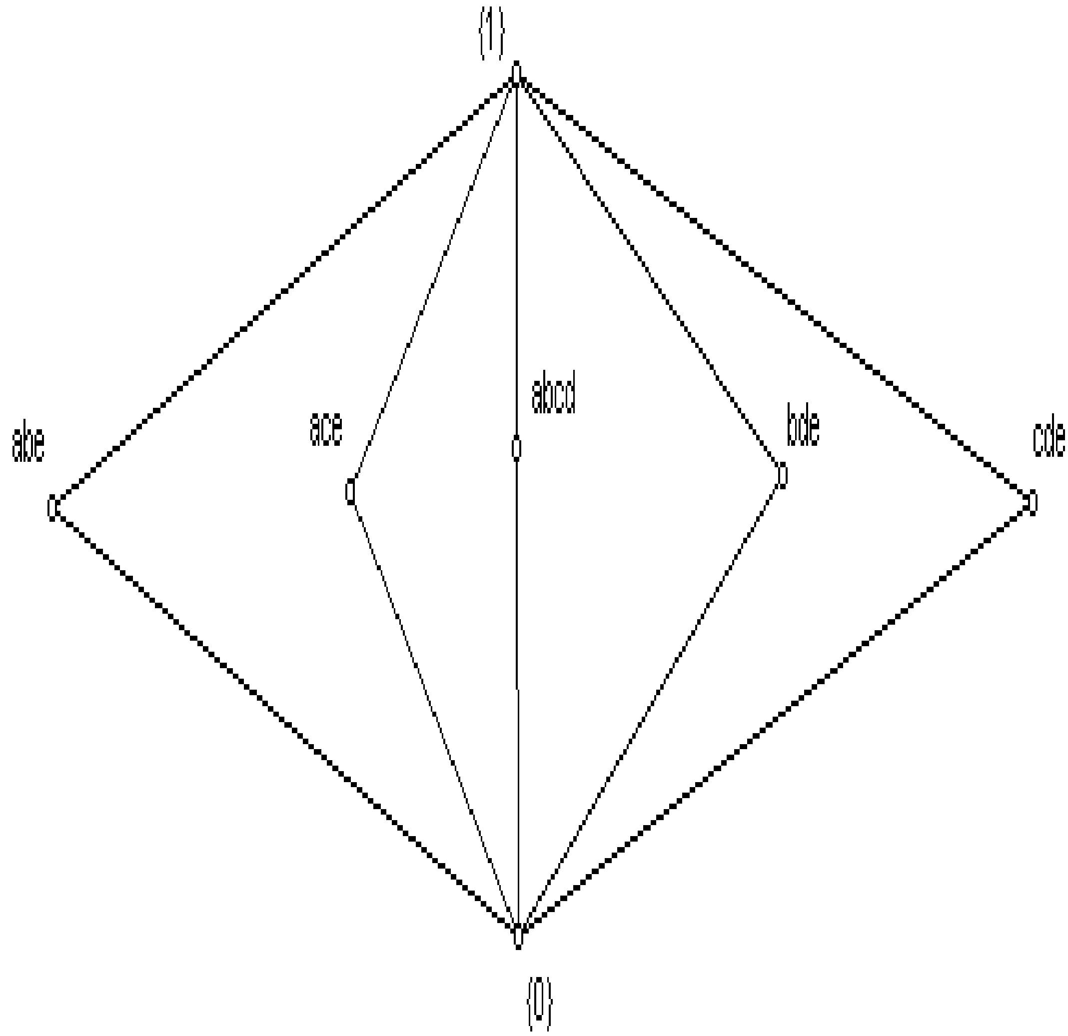
5 maximal chains

$$P_{\{2\}} = f_{\{1\}}$$



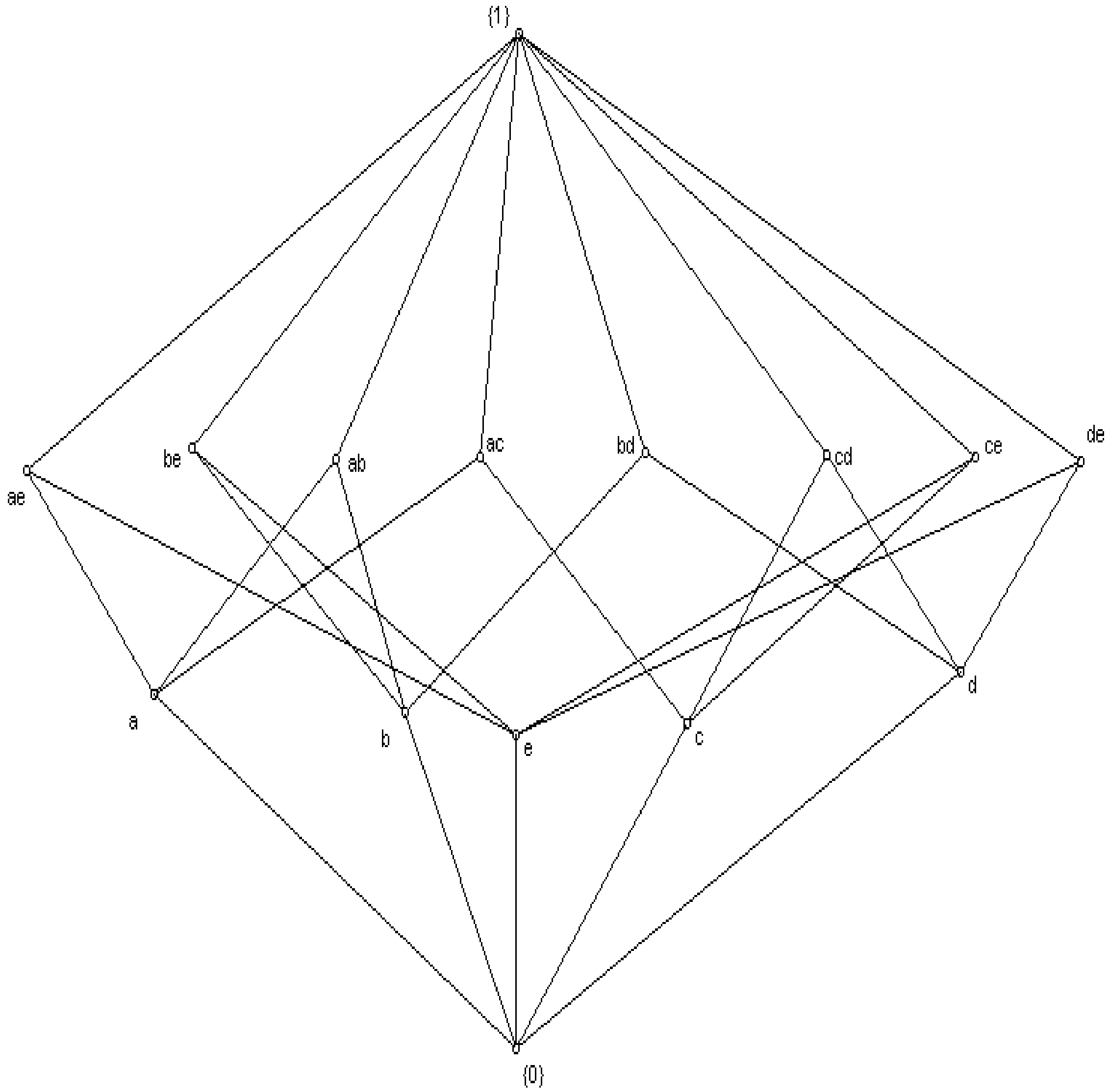
8 maximal chains

$$P_{\{3\}} = f_{\{2\}}$$



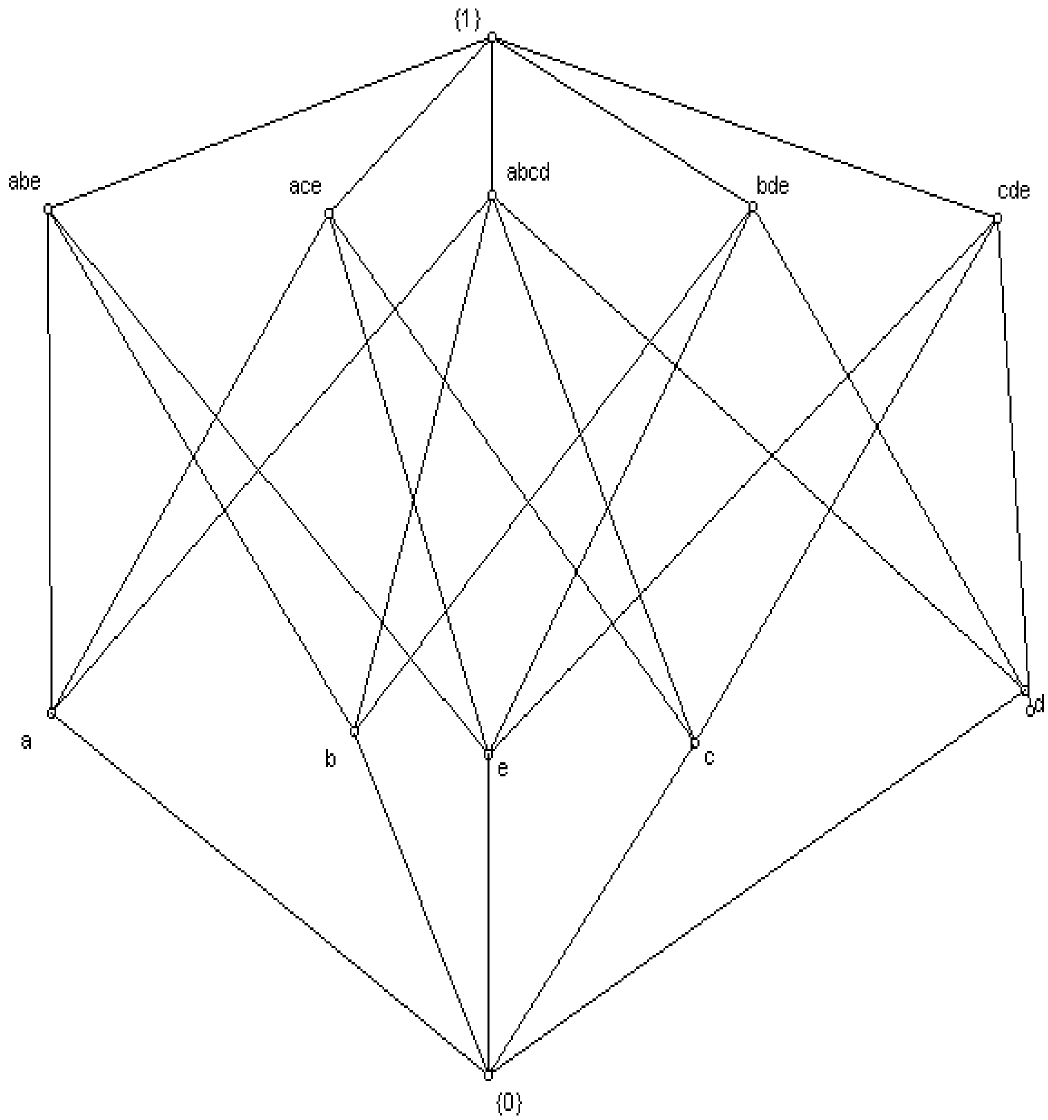
5 maximal chains

$$P_{\{1,2\}} = f_{\{0,1\}}$$



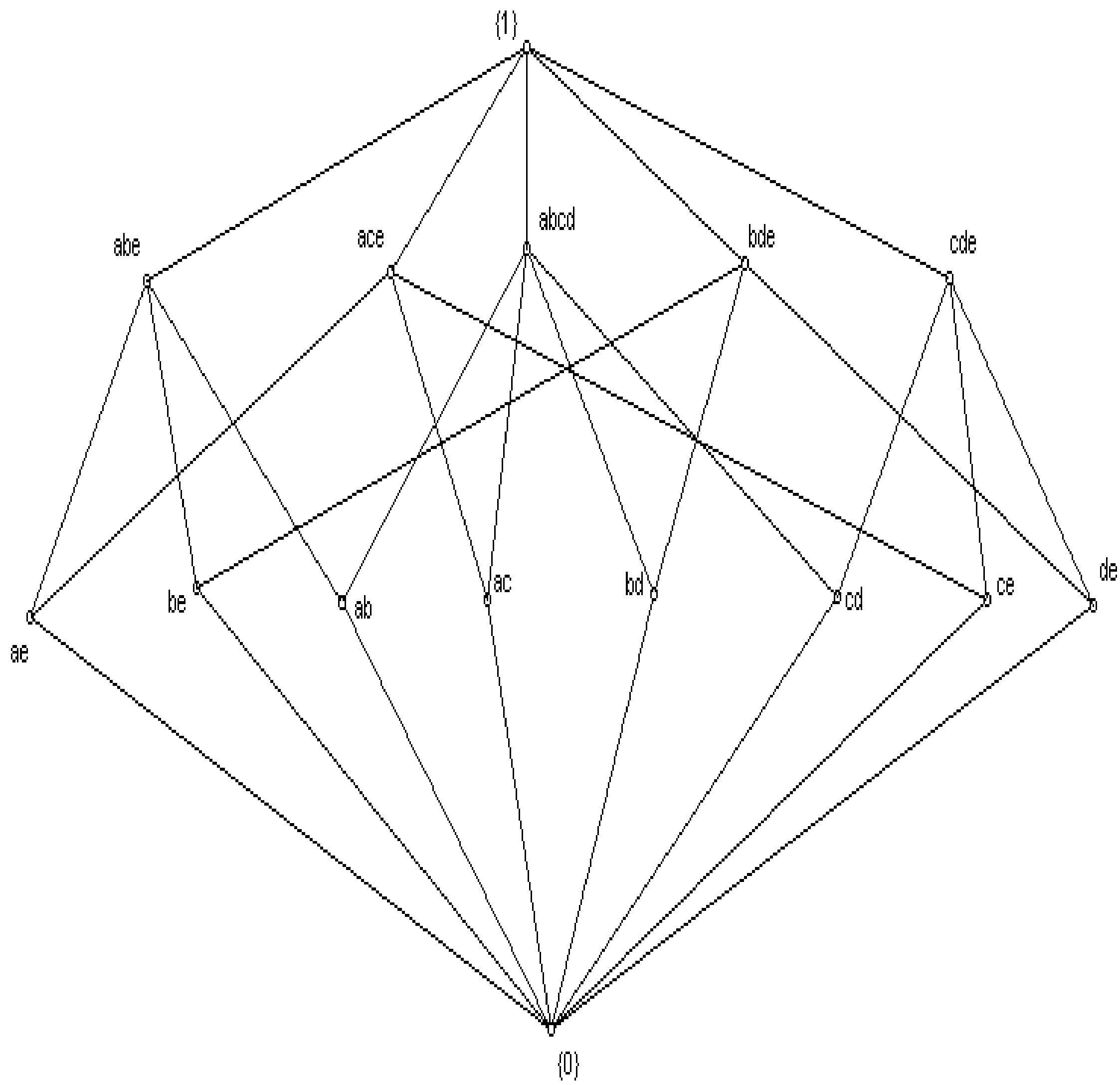
16 maximal chains

$$P_{\{1,3\}} = f_{\{0,2\}}$$



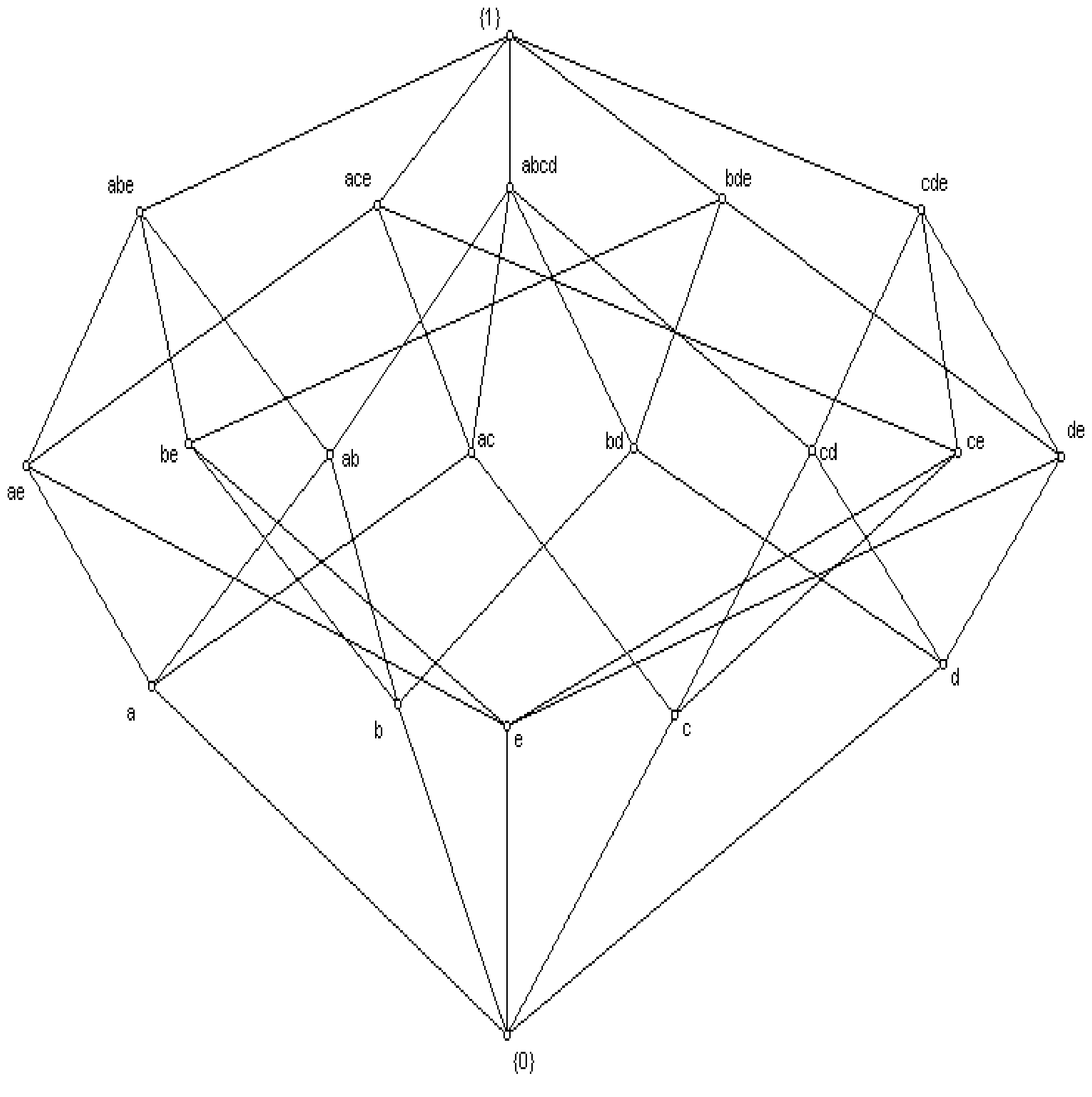
16 maximal chains

$$P_{\{2,3\}} = f_{\{1,2\}}$$



16 maximal chains

$$P_{\{1,2,3\}} = f_{\{0,1,2\}}$$



32 maximal chains

flag h-vector

$$h_S = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T.$$

Example 1. $h_{\{1,2\}} = 1 \cdot f_{1,2} - f_1 - f_2 + f_\emptyset$

Theorem 1 (Stanley). $h_S = h_{\overline{S}}$.

flag h-vector of Egyptian Pyramid

S	f_S	h_S	w_S
$\emptyset$	1	1	
$\{0\}$	5	4	
$\{1\}$	8	7	
$\{2\}$	5	4	
$\{0,1\}$	16	4	
$\{0,2\}$	16	7	
$\{1,2\}$	16	4	
$\{0,1,2\}$	32	1	

$$w_s = w_0 \cdots w_{d-1}$$

$S \subseteq \{0, 1, \ldots, d-1\}$ as above. Write $w_s = w_0 \cdots w_{d-1}$ where

$$w_i = \begin{cases} b, & \text{if } i \in S; \\ a, & \text{if } i \notin S; \end{cases} \tag{1}$$

with a,b noncommutative.

$$w_S = w_0 \cdot w_1 \cdot w_2$$

S	f_S	h_S	w_S
$\emptyset$	1	1	aaa
{0}	5	4	baa
{1}	8	7	aba
{2}	5	4	aab
{0,1}	16	4	bba
{0,2}	16	7	bab
{1,2}	16	4	abb
{0,1,2}	32	1	bbb

ab-index

The ab-index of a polytope is defined as

$$\Psi(P) = \sum_{S \subseteq \{0,1,\ldots,d-1\}} h_s \cdot w_s.$$

S	f_S	h_S	w_S
$\emptyset$	1	1	aaa
$\{0\}$	5	4	baa
$\{1\}$	8	7	aba
$\{2\}$	5	4	aab
$\{0,1\}$	16	4	bba
$\{0,2\}$	16	7	bab
$\{1,2\}$	16	4	abb
$\{0,1,2\}$	32	1	bbb

$\Psi(\text{Pyr}(\Box)) = 1 \cdot aaa + 4 \cdot baa + 7 \cdot aba + 4 \cdot aab + 4 \cdot bba + 7 \cdot bab + 4 \cdot abb + 1 \cdot bbb.$

cd-index

Theorem 2 (Conj. by Fine; Bayer-Billera proved). *The ab-index of a polytope can be written in terms of the non-commutative variables* $c = a + b$ *and* $d = ab + ba$ *called the cd-index.*

Definition 1. deg(c) = 1 and deg(d) = 2

cd-index of Egyptian Pyramid

degree 3 so $ccc = aaa + aba + aab + abb + baa + bba + bab + bbb$

$dc \ = (ab + ba)(ab + ba) = aab + baa + abb + bab$

$cd \ = (a + b)(ab + ba) = aab + bab + aba + bba$

So $\Phi_{\mathsf{Pyr}(\square)}(c, d) = 1 \cdot c^3 + 3 \cdot dc + 3 \cdot cd.$

If degree 4

$c^4, d^2, cdc, c^2d, dc^2.$

Theorem's on cd-index

Theorem 3. *The cd-index of a polygon* P *is given by* $\Phi(P) = c^2 + (f_0 - 2) \cdot d$.

Theorem 4. *The cd-index of a three-dimensional polytope* P *is given by* $\Phi(P) = c^3 + (f_0 - 2) \cdot dc + (f_2 - 2) \cdot cd$.

Theorem 5. *Let* P *be a graded poset. Then* P *has a cd-index with integer coefficients if and only if the h-vector of* P *satisfies the generalized Dehn-Sommerville equations.*

Theorem 6. *If P is Cohen-Macaulay then the h-vector is nonnegative.*

Conjecture 1. Let P be a Gorenstein* poset. Then $\Phi_P(c, d) \geq 0$.